Law of large numbers for the number of nodal surfaces

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1. Introduction

In these notes, we explain Nazarov and Sodin's proof of the law of large numbers for the number of nodal surfaces for very general continuously differentiable Gaussian fields. Given a random subset of $\mathbb{R}^d$, a natural question to ask is "How fast does the number of its connected components contained in a ball grow as a function of the radius?". Here, we will consider zero sets $Z(f) = f^{-1}(\{0\})$ of certain random functions $f : \mathbb{R}^d \to \mathbb{R}$ with Gaussian finite-dimensional marginals.

Before discussing this, let us elaborate on the analogous question (and its answer) in the case of percolation, "How fast does the number of percolation clusters in a box grow as a function of the width of the box?". Let π be a Bernoulli percolation with parameter $p \in [0,1]$ on $\mathbb{Z}^d$ and let K_n be the number of percolation clusters that are completely contained in the box $[-n,n]^d$, then

$$K_n/(2n+1)^d \to \mathbb{E}(1/\#C_0) \quad \text{a.s. and in } L^1 \quad \text{as } n \to \infty,$$

where C_x is the percolation cluster containing $x \in \mathbb{Z}^d$. The proof idea is simple, see [1, Chapter 4] for details. By ergodicity of Bernoulli percolation under shifts, a suitable ergodic theorem implies

$$\frac{1}{(2n+1)^d} \sum_{x \in [-n,n]^d} \frac{1}{\#C_x} \to \mathbb{E}(1/\#C_0) \quad \text{a.s. and in } L^1 \quad \text{as } n \to \infty.$$

We conclude by noting that

$$\sum_{x \in [-n,n]^d} \frac{1}{\#C_x} = K_n + \sum_{x \in [-n,n]^d : C_x \setminus [-n,n]^d \neq \emptyset} \frac{1}{\#C_x} = K_n + O\left(n^{d-1}\right),$$

since $1/\#C_x \leq 1$ for all $x \in \mathbb{Z}^d$ and because the number of clusters intersecting the boundary is clearly bounded by the boundary's size, which is $O\left(n^{d-1}\right)$.

The approach we are considering here will be similar – we will apply an ergodic theorem to well-chosen functions and develop good estimates to control how far the ergodic averages are from the quantities we would like to understand.

From now on, we will consider a centred Gaussian field $f = \left(f(x) : x \in \mathbb{R}^d\right)$ with covariance structure $\mathbb{E}(f(x)f(y)) = K(x,y)$ and $K(x,x) > 0$. The main assumption will be that f is *stationary* i.e. $\left(f(x+v) : x \in \mathbb{R}^d\right)$ and f having the same law for all $v \in \mathbb{R}^d$; this is equivalent to K being of the form $K(x,y) = k(x-y)$. By Bochner's theorem, one deduces that k can be represented as

$$k(x) = \int e^{i\langle x,\zeta\rangle} \rho(d\zeta)$$

for a unique finite Borel measure ρ on $\mathbb{R}^d$ called the *spectral measure* of f.

Without loss of generality, we may restrict ourselves to the case $k(0) = 1$; indeed replacing f by $\left(k(0)^{-1/2} f(x) : x \in \mathbb{R}^d\right)$ will not affect the zero set.

Example 1.1. A good example to keep in mind is the following: For $d = 1$, consider $\zeta_1, ..., \zeta_n \in \mathbb{R}$ and $w_1, ..., w_n \geq 0$. Now define

$$\rho = \sum_i w_i \left(\delta_{\zeta_i} + \delta_{-\zeta_i}\right), \quad k(x) = \sum_i w_i \cos(\zeta_i x)$$
$$\text{and} \quad f(x) = \sum_i \sqrt{w_i}(a_i \cos(\zeta_i x) + b_i \sin(\zeta_i x)),$$

where $(a_i, b_i : i = 1, ..., n)$ are i.i.d. $N(0,1)$ random variables. Then f has covariance k and spectral measure ρ. Intuitively, ρ measures the variances of the individual Fourier modes (and in the stationary case, they all decouple).

It is important to choose a good version of f. A suitable space to consider is

$$C^1_*\left(\mathbb{R}^d\right) = \left\{\alpha \in C^1\left(\mathbb{R}^d\right) : |\alpha(x)| + |\nabla\alpha(x)| > 0 \; \forall x \in \mathbb{R}^d\right\},$$

which carries the σ-algebra generated by $\pi_v : C^1_*\left(\mathbb{R}^d\right) \to \mathbb{R}$ given by $\pi_v(\alpha) = \alpha(v)$ for $v \in \mathbb{R}^d$.

Note that $\alpha \in C^1_*\left(\mathbb{R}^d\right)$ if and only if $\nabla\alpha(x) \neq 0$ for all $x \in Z(\alpha)$ which by the implicit function theorem implies that $Z(\alpha)$ is a codimension 1 submanifold of $\mathbb{R}^d$. We let $N(v, r, \alpha)$ be the number of connected components of $Z(\alpha)$ contained in $B_r(v)$ and $N^\#(v, r, \alpha)$ be the number of connected components of $\partial B_r(v) \setminus Z(\alpha)$. We leave it as an exercise to prove that

$$N : \mathbb{R}^d \times (0, \infty) \times C^1_*\left(\mathbb{R}^d\right) \to \mathbb{N}_0 \quad \text{and} \quad N^\# : \mathbb{R}^d \times (0, \infty) \times C^1_*\left(\mathbb{R}^d\right) \to \mathbb{N}_0$$

are measurable. The aim of these notes is to prove the following (this is a special case of [2, Theorem 1]):

Theorem 1.2 (Nazarov, Sodin). Suppose that the spectral measure satisfies

1. $\int |\zeta|^4 \; \rho(d\zeta) < \infty$,
2. ρ has no atoms,
3. the support of ρ is not contained in any hyperplane.

Then f has a version in $C^1_*\left(\mathbb{R}^d\right)$ and there exists $\nu \in [0, \infty)$ such that

$$\frac{1}{\lambda(B_R(0))} N(0, R, f) \to \nu \quad \text{a.s. and in } L^1 \quad \text{as } R \to \infty.$$

Some remarks on the assumptions: (i) will guarantee the existence of a $C^{2-}\left(\mathbb{R}^d\right)$ version of f, (ii) will imply ergodicity of f under shifts and (iii) will then yield the existence of a $C^1_*\left(\mathbb{R}^d\right)$ version, together with more quantitative results that will be key in establishing certain moment bounds. Also note that $k \in C^4\left(\mathbb{R}^d\right)$ is equivalent to (i), $k(x) \to 0$ as $|x| \to 0$ implies (ii), and rotational invariance of the model implies (iii) (assuming (ii) and k non-constant).

2. Sandwiching the number of nodal surfaces

Lemma 2.1. For $\alpha \in C^1_*\left(\mathbb{R}^d\right)$ and $0 < r < R$, we have

$$\int_{B_{R-r}(0)} \frac{N(x, r, \alpha)}{\lambda(B_r(0))} dx \leq N(0, R, \alpha) \leq \int_{B_{R+r}(0)} \frac{N(x, r, \alpha)}{\lambda(B_r(0))} dx + \int_{B_{R+r}(0)} \frac{N^\#(x, r, \alpha)}{\lambda(B_r(0))} dx.$$

Proof. Let $C_1, ..., C_n$ with $n = N(0, R, \alpha)$ be the connected components of $Z(\alpha)$ contained in $B_R(0)$. Note that for each i,

$$\int_{B_{R-r}(0)} 1(C_i \subset B_r(x))dx \le \lambda(B_r(0)) \le \int_{B_{R+r}(0)} 1(C_i \cap B_r(x) \ne \emptyset)dx.$$

Summing this over i yields

$$\begin{aligned}\int_{B_{R-r}(0)} \frac{N(x,r,\alpha)}{\lambda(B_r(0))}dx &\le N(0,R,\alpha) \\ &\le \int_{B_{R+r}(0)} \frac{N(x,r,\alpha)}{\lambda(B_r(0))}dx + \int_{B_{R+r}(0)} \frac{A(x)}{\lambda(B_r(0))}dx,\end{aligned}$$

where $A(x) = \#\{1 \le i \le n : C_i \not\subset B_r(x), C_i \cap B_r(x) \ne \emptyset\}$. Note that for a.e. $x \in B_{R+r}(0)$, each C_i with $C_i \not\subset B_r(x)$ and $C_i \cap B_r(x) \ne \emptyset$ is nowhere tangent to $\partial B_r(x)$ so that then C_i divides $\partial B_r(x)$ into at least two components; and therefore, since the sets C_i are disjoint, we deduce that $A(x) \le N^{\#}(x, r, \alpha)$. ☐

Both the left- and the right-hand side of the sandwich estimate look promising for an application of an ergodic theorem. Before establishing integrability of the terms in the integrals, we state the ergodic theorem we are going to use, together with a criterion for ergodicity.

3. Ergodic theorem

Theorem 3.1 (Wiener's ergodic theorem). Let $(E, \mathcal{E})$ be a measure space and moreover let $(\tau_v : E \to E : v \in \mathbb{R}^d)$ a family of maps such that $(v, x) \mapsto \tau_v(x)$ is measurable from $\mathbb{R}^d \times E$ to E. Moreover define $\mathcal{J} = \{A \in \mathcal{E} : \tau_v^{-1}(A) = A \ \forall v \in \mathbb{R}^d\}$. Let ξ be a random variable in E s.t. τ_v and ξ have the same law for all $v \in \mathbb{R}^d$. Then for any $\alpha : E \to \mathbb{R}$ with $\alpha(\xi) \in L^1$ we have

$$\frac{1}{\lambda(B_R(0))} \int_{B_R(0)} \alpha(\tau_v \xi)dv \to \mathbb{E}(\alpha(\xi) \mid \xi^{-1}\mathcal{J}) \quad \text{a.s. and in } L^1 \quad \text{as } R \to \infty.$$

Note that in the ergodic case $\xi^{-1}\mathcal{J}$ is (by definition) trivial so that the limit is a constant, namely $\mathbb{E}(\alpha(\xi) \mid \xi^{-1}\mathcal{J}) = \mathbb{E}(\alpha(\xi))$.

Proof. Omitted, see [3, Theorems 2 and 3]. ☐

Theorem 3.2 (Fomin, Grenander, Maruyama). Define $\tau_v : C(\mathbb{R}^d) \to C(\mathbb{R}^d)$ by $\tau_v(\alpha) = \alpha_{\cdot+v}$ and $\pi_v : C(\mathbb{R}^d) \to \mathbb{R}$ by $\pi_v(\alpha) = \alpha_v$ whenever $v \in \mathbb{R}^d$. Consider a set $E \subset C(\mathbb{R}^d)$ which is invariant under τ_v for all $v \in \mathbb{R}^d$ and endow it with $\mathcal{E} = \sigma(\pi_v|_E : v \in \mathbb{R}^d)$. If $f \in E$ is a stationary Gaussian field and its spectral measure ρ has no atoms, then $f^{-1}\mathcal{J}$ is trivial.

Proof. Assume that $A \in \mathcal{E}$ is such that $\tau_v^{-1}(A) = A$ for all $v \in \mathbb{R}^d$, then we need to show that $\mathbb{P}(f \in A) \in \{0, 1\}$. There are $x_k \in \mathbb{R}^d$ (where $k \ge 1$) and functions $\alpha_k : \mathbb{R}^k \to [0, 1]$ such that $\alpha_k(\pi_{x_1}, ..., \pi_{x_k}) \uparrow 1_A$ as $k \to \infty$. Then

$$\begin{aligned}\mathbb{P}(f \in A) &= \mathbb{P}(f \in A, \tau_v f \in A) \\ &\le \mathbb{E}\left(\alpha_k\left(f_{x_1}, ..., f_{x_k}\right)\alpha_k\left(f_{x_1+v}, ..., f_{x_k+v}\right)\right) \\ &\quad +2\mathbb{E}\left|1(f \in A) - \alpha_k\left(f_{x_1}, ..., f_{x_k}\right)\right|.\end{aligned}$$

It suffices to show $\liminf_{|v| \to\infty} \operatorname{cov}\left(\alpha_k\left(f_{x_1}, ..., f_{x_k}\right), \alpha_k\left(f_{x_1+v}, ..., f_{x_k+v}\right)\right) = 0$ since this implies

$$\begin{aligned}\mathbb{P}(f \in A) &\leq \mathbb{E}\left(\alpha_k\left(f_{x_1}, ..., f_{x_k}\right)\right)^2 + 2\mathbb{E}\left|1(f \in A) - \alpha_k\left(f_{x_1}, ..., f_{x_k}\right)\right| \\ &\to \mathbb{P}(f \in A)^2 \quad \text{as} \quad k \to \infty,\end{aligned}$$

hence $\mathbb{P}(f \in A) \in \{0, 1\}$. To see this, observe that for $c = \max_{ij} |x_i - x_j|$,

$$\begin{aligned}&\frac{1}{\lambda(B_R(0))} \int_{B_R(0)} \sum_{ij} k\left(x_i - x_j + v\right)^2 dv \\ &\leq \frac{k^2}{\lambda(B_R(0))} \int_{B_{R+c}(0)} k(x)^2 dx \to k^2 \sum_{\zeta\in\mathbb{R}^d} \rho(\{\zeta\})^2 = 0\end{aligned}$$

as $R \to \infty$ where the convergence above follows from Wiener's lemma. Therefore $\liminf_{|v| \to\infty} \sum_{ij} k\left(x_i - x_j + v\right)^2 = 0$ and the result follows. □

4. Versions and moment bounds

By assumption (i), the covariance function k is in $C^4\left(\mathbb{R}^d\right)$. By Kolmogorov's extension theorem, this implies that f has a $C^{2-}\left(\mathbb{R}^d\right)$ version and (switching to such a version), the Hölder norm

$$\|f\|_{B_r(x),1+\beta} = \sup_{B_r(x)} |f| + \sup_{B_r(x)} |\nabla f| + \sup_{v,w\in B_r(x):v\neq w} \frac{|\nabla f(v) - \nabla f(w)|}{|v - w|^\beta}$$

has a Gaussian tail and hence moments of all orders $p < \infty$ for $r > 0$ and $x \in \mathbb{R}^d$ whenever $\beta \in (0, 1)$. The key in the proofs of this section will be the definition of the following functions:

$$\begin{aligned}\Phi(x) &= |f(x)|^{-t} |\nabla f(x)|^{-td} \\ \Psi(x) &= |f(x)|^{-t}\ |\nabla_S f(x)|^{-t(d-1)} \quad (x \neq 0).\end{aligned}$$

Here, $\nabla_S f(x) = \nabla f(x) - \langle\nabla f(x), x\rangle x/|x|^2$ is the projection of $\nabla f(x)$ onto the plane perpendicular to x. Let us make the following observation.

Lemma 4.1. For each $x \in \mathbb{R}^d$, $f(x)$ and $\nabla f(x)$ are independent and the law of $(f(x), \nabla f(x))$ is non-degenerate. Moreover, for $tp < 1$, $\mathbb{E}\Phi(0)^p, \mathbb{E}\Psi(0)^p < \infty$.

Proof. We have $\mathbb{E}(f(x)\partial_i f(x)) = \partial_i k(0)$ but by definition, $k(x) = k(-x)$ and hence $\partial_i k(0) = 0$. By independence, it is enough to show that $\nabla f(x)$ is non-degenerate (recall that $f(x) \sim N(0, 1)$). Suppose not, then since it is Gaussian, there exists $v \in \mathbb{R}^d \setminus \{0\}$ such that $\langle\nabla f(x), v\rangle = 0$ a.s. and hence

$$0 = \mathbb{E}\langle\nabla f(x), v\rangle^2 = -\sum_{ij} v_i v_j \partial_{ij} k(0) = \int \langle v, \zeta\rangle^2 \rho(d\zeta)$$

so that the support of ρ is contained in the hyperplane perpendicular to v which contradicts assumption (iii). The last claim follows from $\mathbb{E}\ |X|^{-\alpha} < \infty$ for $X \sim N(0, I_n)$ and $\alpha < n$ combined with a change of variables. □

Lemma 4.2. Almost surely $|f(x)| + |\nabla f(x)| > 0$ for all $x \in \mathbb{R}^d$.

Proof. Fix $R > 0$. For $t < 1$ we get

$$\infty > R^d \mathbb{E}\Phi(0) = \mathbb{E}\int_{B_R(0)} \Phi(x)dx = \mathbb{E}\int_{B_R(0)} |f(x)|^{-t}|\nabla f(x)|^{-td}\ dx.$$

If $f(x_0) = 0$ and $\nabla f(x_0) = 0$ at some point $x_0 \in B_R(0)$, then

$$\int_{B_R(0)} |f(x)|^{-t}|\nabla f(x)|^{-td}\ dx \geq \int_{B_R(0)} \|f\|_{B_R(0),1+\beta}^{-t(1+d)}|x - x_0|^{-t(1+d\beta)}\ dx.$$

Take $t, \beta \in (0,1)$ so that $d > t(1 + d\beta)$; then the last integral diverges and hence a.s. there is no such point x_0. □

Proposition 4.3. There exists a constant $C > 0$ such that for all $R \geq 1$, $\mathbb{E}N(0,R,f) \leq CR^d$ and $\mathbb{E}N^{\#}(0,R,f) \leq CR^{d-1}$.

Proof. We first prove $\mathbb{E}N(0,R,f) \leq CR^d$. Let $C_1, ..., C_n$ be the connected components of $B_R(0) \setminus Z(f)$ which have their closure contained in $B_R(0)$. Then $N(0,R,f) \leq n$. Pick $z_i \in C_i$ such that $\nabla f(z_i) = 0$ and let $r_i = d(z_i, \partial C_i)$. Then $B_{r_i}(z_i) \subset C_i$ and $\partial B_{r_i}(z_i) \cap \partial C_i \ni x_i$, say. Thus

$$|f(x)| \leq \|f\|_{B_{r_i}(z_i),1+\beta}\ |x - x_i| \quad \text{and} \quad |\nabla f(x)| \leq \|f\|_{B_{r_i}(z_i),1+\beta}\ |x - z_i|^\beta$$

for $x \in B_{r_i}(z_i)$. If $r_i \leq 1$,

$$\begin{aligned}\int_{B_{r_i}(z_i)} \Phi = \int_{B_{r_i}(z_i)} \frac{dx}{|f(x)|^t\ |\nabla f(x)|^{td}} &\geq \|f\|_{B_{r_i}(z_i),1+\beta}^{-t(1+d)} \int_{B_{r_i}(z_i)} \frac{dx}{|x - x_i|^t |x - z_i|^{td\beta}} \\ &\geq C\ \|f\|_{B_{r_i}(z_i),1+\beta}^{-t(1+d)}\ r_i^{d-t(1+d\beta)}\end{aligned}$$

for a constant $C > 0$.

Choose $t, \beta \in (0,1)$ with $d - t(1 + d\beta) \leq 0$, which implies $r_i^{d-t(1+d\beta)} \geq 1$. Moreover, the number of i for which $r_i > 1$ is clearly bounded by $\lambda(B_R(0))/\lambda(B_1(0)) = R^d$, so that by summing over i we get

$$\begin{aligned}N(0,R,f) \leq n &\leq R^d + C^{-1} \sum_{i:r_i\leq 1} \|f\|_{B_{r_i}(z_i),1+\beta}^{t(1+d)} \int_{B_{r_i}(z_i)} \Phi \\ &\leq R^d + C^{-1} \sum_{i:r_i\leq 1} \int_{B_{r_i}(z_i)} \Phi(x)\ \|f\|_{B_{2r_i}(x),1+\beta}^{t(1+d)}\ dx \\ &\leq R^d + C^{-1} \int_{B_R(0)} \Phi(x)\ \|f\|_{B_2(x),1+\beta}^{t(1+d)}\ dx.\end{aligned}$$

By taking expectations, using stationarity and observing

$$\mathbb{E}\Big(\Phi(0)\ \|f\|_{B_2(0),1+\beta}^{t(1+d)}\Big) \leq \mathbb{E}(\Phi(0)^p)^{1/p}\mathbb{E}\Big(\|f\|_{B_2(0),1+\beta}^{tq(1+d)}\Big)^{1/q} < \infty$$

for $p \in (1, 1/t)$ and $1/p + 1/q = 1$, we obtain the claim.

The proof of the second assertion is entirely analogous: There, one lets $C_1, ..., C_n$ with $n = N^{\#}(0,R,f)$ be the connected components of $\partial B_R(0) \setminus Z(f)$, takes $z_i \in C_i$ such that $\nabla_S f(z_i) = 0$ (e.g. by taking z_i such that $|f(z_i)| = \sup_{C_i} |f|$), defines $r_i = d(z_i, \partial C_i)$ and chooses $x_i \in \partial B_{r_i}(x_i) \cap \partial C_i$ with ∂C_i being the boundary of C_i within the sphere $\partial B_R(0)$. Let us write σ for the (non-normalised) surface measure supported on $\partial B_R(0)$. We note that for $x, y \in B_{r_i}(z_i)$,

$$
\begin{aligned}
|\nabla_S f(x) - \nabla_S f(y)| &\le |\nabla f(x) - \nabla f(y)| + \left|\langle x, \nabla f(x)\rangle \frac{x}{R^2} - \langle y, \nabla f(y)\rangle \frac{y}{R^2}\right| \\
&\le \|f\|_{B_{r_i}(z_i),1+\beta} \left(2 \cdot |x-y|^\beta + \frac{2}{R} \cdot |x-y|\right) \\
&\le C\|f\|_{B_{r_i}(z_i),1+\beta} \; |x-y|^\beta
\end{aligned}
$$

for some universal constant $C > 0$. Then for $r_i \le 1$,

$$
\begin{aligned}
\int_{B_{r_i}(z_i)} \Psi(x)\sigma(dx) &= \int_{B_{r_i}(z_i)} \frac{\sigma(dx)}{|f(x)|^t \; |\nabla_S f(x)|^{t(d-1)}} \\
&\ge C^{-t(d-1)} \|f\|_{B_{r_i}(z_i),1+\beta}^{-td} \int_{B_{r_i}(z_i)} \frac{\sigma(dx)}{|x-x_i|^t |x-z_i|^{t\beta(d-1)}} \\
&\ge C' \; \|f\|_{B_{r_i}(z_i),1+\beta}^{-td} \; r_i^{d-1-t(1+\beta(d-1))}
\end{aligned}
$$

for some $C' > 0$. Choose $t, \beta \in (0,1)$ such that $d - 1 - t(1+\beta(d-1)) \le 0$. Again, we note that the number of i with $r_i > 1$ is bounded by $\sigma(\partial B_R(0))/\sigma(B_1(R,0,...,0)) = O(R^{d-1})$. Therefore, for some constant $C'' > 0$,

$$
N^{\#}(0,R,f) = n \le C'' \left(R^{d-1} + \sum_{i: r_i \le 1} \|f\|_{B_{r_i}(z_i),1+\beta}^{td} \int_{B_{r_i}(z_i)} \Psi(x)\sigma(dx) \right)
$$

and one concludes the proof in the same way as in the first case. □

5. Proof of the main theorem

Proof. From the previous section, we know that f has a $C^1_*(\mathbb{R}^d)$ version, that $N(0,r,f)$ and $N^{\#}(0,r,f)$ are integrable for each $r > 0$ and that

$$
\frac{\mathbb{E}N(0,r,f)}{\lambda(B_r(0))} = O(1) \quad \text{and} \quad \frac{\mathbb{E}N^{\#}(0,r,f)}{\lambda(B_r(0))} = O\left(\frac{1}{r}\right) \tag{1}
$$

as $r \to \infty$. In particular, we can take a sequence $r_k \to \infty$ such that $\lambda\left(B_{r_k}(0)\right)^{-1}\mathbb{E}N(0,r_k,f) \to \nu$ as $k \to \infty$ for some $\nu \in [0,\infty)$. Using the sandwich estimate, we get

$$
\begin{aligned}
\left|\frac{N(0,R,f)}{\lambda(B_R(0))} - \nu\right| &\le \left|\frac{N(0,R,f)}{\lambda(B_R(0))} - \frac{\mathbb{E}N(0,r,f)}{\lambda(B_r(0))} \;|+|\; \frac{\mathbb{E}N(0,r,f)}{\lambda(B_r(0))} - \nu\right| \\
&\le \left|\frac{\left(1-\frac{r}{R}\right)^d}{\lambda(B_{R-r}(0))} \int_{B_{R-r}(0)} \frac{N(x,r,f)}{\lambda(B_r(0))} dx - \frac{\mathbb{E}N(0,r,f)}{\lambda(B_r(0))}\right| \\
&\quad + \left|\frac{\left(1+\frac{r}{R}\right)^d}{\lambda(B_{R+r}(0))} \int_{B_{R+r}(0)} \frac{N(x,r,f)}{\lambda(B_r(0))} dx - \frac{\mathbb{E}N(0,r,f)}{\lambda(B_r(0))}\right| \\
&\quad + \left|\frac{\left(1+\frac{r}{R}\right)^d}{\lambda(B_{R+r}(0))} \int_{B_{R+r}(0)} \frac{N^{\#}(x,r,f)}{\lambda(B_r(0))} dx - \frac{\mathbb{E}N^{\#}(0,r,f)}{\lambda(B_r(0))}\right| \\
&\quad + \left|\frac{\mathbb{E}N(0,r,f)}{\lambda(B_r(0))} - \nu\right| + \frac{\mathbb{E}N^{\#(0,r,f)}}{\lambda(B_r(0))}.
\end{aligned}
$$

For fixed $r > 0$, the first three terms on the right-hand side of the estimate tend to 0 a.s. and in L^1 as $R \to \infty$ by our ergodic theorem; indeed $N(x, r, f) = N(0, r, \tau_v f)$ and $N^{\#}(x, r, f) = N^{\#}(0, r, \tau_v f)$, also integrability follows from (1), ergodicity from assumption (i) and finally it is easy to control the effect of the $(1 \pm r/R)^d$ terms (separately in the a.s. and L^1 case).

Also, by taking $r = r_k$, the last two terms tend to 0 as $k \to \infty$ (uniformly in R, trivially). Both in the a.s. and in the L^1 case, it is easy to combine the two results above to finish the proof. □

Remark 5.1. The sandwich estimate also holds with $N(0, R, \alpha)$ replaced by the number of connected components of $Z(\alpha)$ intersecting $B_R(0)$ and one can then easily check that the analogous law of large numbers (also a.s. and in L^1) holds with the same limit $\nu \geq 0$.

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