Derivative pricing basics

Matthis Lehmkuehler

24/07/2023

1. Introduction

Derivative pricing is an important application of stochastic calculus and in this short note, we are going to explain how to do so using replicating portfolios. We will assume that the reader is fluent in stochastic analysis but see any good textbook like [1] or [2] on the topic.

We focus here on the very simplest setting and refer the reader to [3] for a very nice introduction into more complicated cases – the formalism however remains identical. In this note, for the sake of conciseness, we ignore technical details somewhat but they are not hard to add.

Let (W_t) be a standard Brownian motion, let $(\mathcal{F}_t)$ be the filtration it generates and let us suppose that the asset price follows the SDE $dS_t = S_t(\mu dt + \sigma dW_t)$ and we have a bond $dB_t = rB_t dt$ where $r \in \mathbb{R}$ is the interest rate, $\mu \in \mathbb{R}$ is the bias for the stock price and $\sigma > 0$ is the volatility.

Suppose that V_T^* is a derivative which is measurable with respect to $\mathcal{F}_T$ for some time horizon $T \geq 0$ and we wish to determine its fair price. The idea of finding the fair price is via a replicating portfolio and goes as follows:

- We want to find adapted and continuous processes (a_t) and (b_t) such that $V_t := a_t S_t + b_t B_t$ satisfies $V_T = V_T^*$. We call $((a_t), (b_t))$ the replicating portfolio since we are replicating the derivative by holding certain amounts of stocks and bonds to model the derivative. The adaptedness condition exactly captures that the replicating strategy does not look into the future which is of course crucial.
- We need the replicating portfolio to be self-financing. For this to be the case, we intuitively need that $a_t S_{t+\delta} + b_t B_{t+\delta} = a_{t+\delta} S_{t+\delta} + b_{t+\delta} B_{t+\delta}$ and this yields on a more formal level the self-financing condition

$$dV_t = a_t dS_t + b_t dB_t. \tag{1}$$

- The fair price is then given by $V_0 = a_0 S_0 + b_0 B_0$. Indeed, if the price was anything else, then this would yield an arbitrage opportunity because two equivalent assets, the derivative and the replicating portfolio, would be offered at different prices on the market.

There are two approaches to finding (V_t). One goes via an explicit Itô calculation and yields the Black-Scholes equation, the other one goes via a clever insight into performing a change of measure (and is the starting point to the theory of equivalent martingale measures).

2. Black-Scholes equation

Suppose now that $V_t = V(t, S_t)$ for some function V. Then we can use Itô's formula to compute

$$dV(t,S_t) = \left(\partial_t V(t,S_t) + \mu S_t \partial_s V(t,S_t) + \left(\sigma^2 \frac{S_t^2}{2}\right)\partial_{ss} V(t,S_t)\right)dt$$
$$+\sigma S_t \partial_s V(t,S_t) dW_t$$

and by comparing terms with the self-financing condition (1), we get $a_t = \partial_s V(t, S_t)$ and

$$b_t r B_t + a_t \mu S_t = \partial_t V(t, S_t) + \mu S_t \partial_s V(t, S_t) + \left(\sigma^2 \frac{S_t^2}{2}\right) \partial_{ss} V(t, S_t)$$

Using that $b_t B_t = V(t, S_t) - a_t S_t$ and $a_t = \partial_s V(t, S_t)$ we are thus led to the Black-Scholes equation

$$\partial_t V(t, s) + rs \partial_s V(t, s) + (\sigma^2 s^2 / 2) \partial_{ss} V(t, s) = r V(t, s).$$

The boundary condition we specify is $s \mapsto V(T, s)$. After solving the PDE, we can read off the fair price and it is given by $V(0, S_0)$.

3. Equivalent martingale measures

The fact that there is no dependence on μ in the Black-Scholes formula is not surprising since the computation is robust under Girsanov changes of measure.

We perform a Girsanov change of measure such that under the new measure $\mathbb{Q}$, the process $(e^{-rt} A_t : t \leq T)$ is a martingale for all assets A, so for $A \in \{S, B\}$ in our setting. Note that the martingale property trivially holds for B since $(e^{-rt} B_t)$ is a constant process (and this is in fact what forces us to take the e^{-rt} factor here).

Recall that $V_t = a_t S_t + b_t B_t$ and $dV_t = a_t dS_t + b_t dB_t$. We claim that $(e^{-rt} V_t : t \leq T)$ is also a martingale. Indeed,

$$\begin{aligned} d(e^{-rt} V_t) &= e^{-rt} dV_t - r e^{-rt} V_t dt \\ &= e^{-rt} (a_t dS_t + b_t dB_t) - r e^{-rt} (a_t S_t + b_t B_t) dt \\ &= a_t (e^{-rt} dS_t - r e^{-rt} S_t dt) + b_t (e^{-rt} dB_t - r e^{-rt} B_t dt) \\ &= a_t d(e^{-rt} S_t) + b_t d(e^{-rt} B_t) \end{aligned}$$

implies that $(e^{-rt} V_t : t \leq T)$ is a martingale. Therefore by the martingale property we get the fair price

$$V_0 = \mathbb{E}_{\mathbb{Q}}(e^{-rT} V_T) = e^{-rT} \mathbb{E}_{\mathbb{Q}}(V_T^*). \tag{2}$$

If we let

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\frac{r - \mu}{\sigma} W_T - \frac{(r - \mu)^2}{2\sigma^2} T\right)$$

then by Girsanov's theorem under $\mathbb{Q}$ the process $(W_t : t \leq T)$ is a standard Brownian motion with drift $(r - \mu)/\sigma$, say $W_t = W_t' + t(r - \mu)/\sigma$ for $t \leq T$ where (W_t') is a standard Brownian motion (with no drift) under $\mathbb{Q}$. Then

$$S_t = S_0 e^{\sigma W_t' + (r - \sigma^2/2)t} \quad \text{and} \quad B_t = B_0 e^{rt}$$

for $t \leq T$ and using (2) we can then simply compute expectations in order to find the fair price V_0 for the derivative with payout V_T^* at time T.

The entire story is easily modifiable for instance to the case of a model with dividends, i.e. in a small time interval dt we are being paid a dividend $\delta a_t S_t dt$. In this case, we can either change the self-financing condition to $dV_t = a_t dS_t + b_t dB_t + \delta a_t S_t dt$ or simply say that if we define a new asset S' where all dividends are instantaneously reinvested, we have $S_t' = e^{\delta t} S_t$ and we can take the above normal analysis and directly apply it there.

References

[1] J.-F. Le Gall, *Brownian Motion, Martingales, and Stochastic Calculus*. Springer Publishing Company, 2018. doi: 10.1007/978-3-319-31089-3

[2] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*. Springer Berlin Heidelberg, 2004. doi: 10.1007/978-3-662-06400-9

[3] M. Baxter and A. Rennie, *Financial Calculus: An Introduction to Derivative Pricing*. Cambridge University Press, 1996. doi: 10.1017/CBO9780511806636